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ONE RELATOR GROUPS ARE SEMISTABLE AT INFINITY

MICHAEL L. MIHALIK and STEVEN T. TSCHANTZ

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IN THIS paper, we show that all finitely generated one relator groups satisfy a homotopy property. For a connected CW-complex Y proper rays $r, s: [0, \infty) \to Y$ *converge to the same end of* Y if for any compact set $C \subseteq Y$ there is an integer N such that $r([N, \infty))$ and $s([N, \infty))$ are contained in the same component of $Y - C$. The space Y is *semistable at infinity* if any two proper rays converging to the same end of Y are properly homotopic. A finitely presented group G is *semistable at infinity* if for some (equivalently any) finite complex X with $\pi_1(X) = G$ the universal cover of X is semistable at infinity. Our main result is

THEOREM 1. *All finitely generated one relator groups are semistable at infinity.*

By [1] we have

COROLLARY 2. *For any one relator group G, $H^2(G; \mathbb{Z}G)$ is free Abelian.*

Our main tool in proving Theorem 1 is the following result from [3].

THEOREM 3. *If $G = A *_H B$ is an amalgamated product where A and B are finitely presented and semistable at infinity, and H is finitely generated then G is semistable at infinity. If $G = A *_H$ is an HNN-extension where A is finitely presented and semistable at infinity and H is finitely generated then G is semistable at infinity.*

Proof of Theorem 1. We use the following structure theorem for one relator groups patterned after Magnus' proof of the Freiheitssatz (see [2]).

LEMMA 4. *Given any finitely generated one relator group G there exists a finite sequence of finitely generated one relator groups $H_1, H_2, \ldots H_n = G$ such that for each $i < n$, either H_{i+1} or $H_{i+1} * \mathbb{Z}$ is an HNN-extension of H_i over a finitely generated group, and such that H_1 is either a free group or else is a free product of a free group and a finite cyclic group.*

The proof of the theorem also uses the following fact.

LEMMA 5. *If G is finitely presented and $G * \mathbb{Z}$ is semistable at infinity then G is semistable at infinity.*

Then each H_i is seen to be semistable at infinity by induction on i. Since free groups and finite groups are semistable at infinity, H_1 is semistable at infinity. If H_i is semistable at infinity then, by Theorem 3, an HNN-extension of H_i over a finitely generated group is semistable at infinity, so either H_{i+1} or $H_{i+1} * \mathbb{Z}$ is semistable at infinity. In the later case, by

Lemma 5, H_{i+1} is semistable at infinity again. But then by induction, each H_i is semistable at infinity and so $G = H_n$ is semistable at infinity. ∎

Proof of Lemma 4. The proof is by induction on the length N of a cyclically reduced relator r defining G and for each fixed N by induction on $N - k$ where k is the number of generators appearing in r. If $N = 0$ then G is a free group. If $N > 0$ and $N - k = 0$, i.e. each generator in r appears exactly once, then G is a free group with one fewer generator. Otherwise, it will suffice to find a finitely generated one relator group H with defining relator r' of length N' involving k' generators such that either $N' < N$ or else $N' = N$ and $N' - k' < N - k$, and such that either G or $G * \mathbb{Z}$ is an HNN-extension of H over a finitely generated group.

First suppose that some letter a appearing in r has zero exponent sum in r. For each other generator b appearing in r define $b_i = a^{-i}ba^i$ and rewrite r as a product r' of the b_i (different b_i corresponding to each different b). Since there is at least one occurrence of a in r and each other b is rewritten as a single b_i, r' is shorter than r. For each b in r, let m_b and M_b be the greatest lower bound and least upper bound on the i with b_i appearing in r'. Let H have as generators the set of all b_i, $m_b \leq i \leq M_b$, together with any generators of G not appearing in r (so a is excluded) and single relator r'. Let K be the subgroup of H generated by all of the b_i, $m_b \leq i < M_b$. Then K is a free group freely generated by these b_i since the generators of K omit one of the generators of H appearing in r'. Similarly, let K' be the subgroup of H generated by all of the b_i, $m_b < i \leqslant M_b$, so that K' is also free and hence isomorphic to K by the isomorphism mapping b_i to b_{i+1}. But then the HNN-extension of H by these isomorphic subgroups with stable letter a has presentation $\langle a, b, \ldots : r', \; a^{-1}b_i a = b_{i+1} (m_b \leq i < M_b)\rangle$. Defining additional b_i as needed, we can derive $b_i = a^{-i}b_0 a^i$ and so defining $b = b_0$ we get back the presentation of G, i.e. G is an HNN-extension of H and this case is complete.

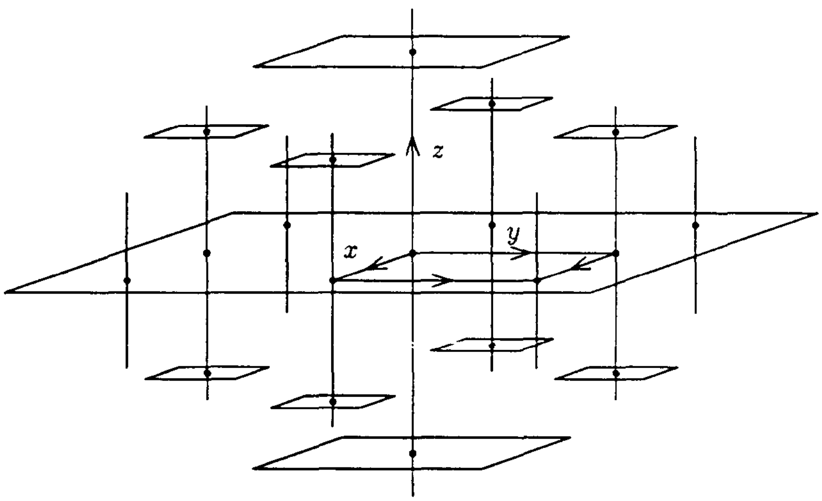
Suppose now that no letter appears in r with zero exponent sum. If only one letter appears in r we have a group isomorphic to a free product of a free group and $\mathbb{Z}_N$. If more than one letter appears in r then there will be two cases depending on whether or not there are distinct letters a and b in r satisfying a certain condition. If such a and b exist then we construct a one relator group H with relator of the same length as r but involving more generators, such that $G * \mathbb{Z}$ is an HNN-extension of H over a finitely generated group. The condition on a and b will be devised just so this construction works. If no a and b can be found satisfying the condition then we show that G is a free product of a free group and a finite cyclic group.

Let x be a letter distinct from the generators of G so $G' = G * \mathbb{Z}$ has presentation $\langle a, b, \ldots, x : r\rangle$. Let a and b be distinct letters appearing in r and let $\alpha = \sigma_a(r)$ and $\beta = \sigma_b(r)$ (where as usual $\sigma_a(r)$ denotes the exponent sum of the letter a in r.) Add new generators to this presentation with defining relations $y = ax^\beta$ and $z = bx^{-\alpha}$. Solve for a and b in terms of y and z, substitute $a = yx^{-\beta}$ and $b = zx^\alpha$ back into r to get a word r_1, and eliminate the now redundant a and b. Hence G' has presentation $\langle y, z, \ldots, x : r_1\rangle$ where y and z have replaced a and b in the list of generators and where now $\sigma_x(r_1) = \sigma_a(r)(-\beta) + \sigma_b(r)\alpha = 0$. Proceed as in the previous case to rewrite r_1 as a word r_1' in $c_i = x^{-i}cx^i$ where c ranges over the letters appearing in r_1 except x. Now r_1' will have the same length as r since each a becomes a y_i, each b becomes a z_i and each other c becomes a c_i in r_1'. Taking $H = \langle y_i, \ldots z_i \ldots : r_1'\rangle$ we get that $G' = G * \mathbb{Z}$ is an HNN-extension of H over a finitely generated (free) group with stable letter x. If r_1' involves more generators than r then we take $r' = r_1'$ and we are done (since $N' = N$ and $N' - k' < N - k$). We must examine this rewriting procedure in more detail to see exactly when this is the case.

Each occurrence of a letter c in r corresponds to a triple (s, u, t) where $u = c^{\pm 1}$, and $r = sut$. Define $I_{a,b}(s, u, t)$ for distinct letters a and b, $u = c^{\pm 1}$ for a letter c, and words s and t as follows. If $u \neq a^{-1}$, b^{-1}, let $I_{a,b}(s, u, t) = \sigma_a(s)\sigma_b(sut) - \sigma_b(s)\sigma_a(sut)$. Let $I_{a,b}(s, a^{-1}, t) = (\sigma_a(s) - 1)\sigma_b(sut) - \sigma_b(s)\sigma_a(sut)$, and $I_{a,b}(s, b^{-1}, t) = \sigma_a(s)\sigma_b(sut) - (\sigma_b(s) - 1)\sigma_a(sut)$. If $r = sut$ and $u = c^{\pm 1}$ then $\sigma_a(sut) = \alpha$, $\sigma_b(sut) = \beta$, and $I_{a,b}(s, u, t)$ is the negative of the exponent sum of x in the initial segment of r_1 before the corresponding occurrence of u in r_1 (or if $u = a^{-1}$ or b^{-1}, the negative of the exponent sum of x in the initial segment of r_1 before the corresponding y^{-1} or z^{-1}), and so this occurrence of $c^{\pm 1}$ in r becomes a $c_i^{\pm 1}$ in r_1' where $i = I_{a,b}(s, u, t)$ (or $y_i^{\pm 1}$ or $z_i^{\pm 1}$ if $u = a^{\pm 1}$ or $b^{\pm 1}$). Thus more generators appear in r_1' than appear in r provided that for some occurrences of a letter c in $r = sut = s'u't'$, $u = c^{\pm 1}$ and $u' = c^{\pm 1}$, we have $I_{a,b}(s, u, t) \neq I_{a,b}(s', u', t')$.

Now it may happen that for a particular a and b, r_1' turns out to be just a relabeling of r and yet for a different choice of a and b, r_1' would involve more generators than r and we can proceed as outlined above. Suppose instead that for every pair of distinct generators a and b appearing in r and for every pair of occurrences of c in $r = sut = s'u't'$, $u = c^{\pm 1}$ and $u' = c^{\pm 1}$ we have $I_{a,b}(s, u, t) = I_{a,b}(s', u', t')$. We now show that in this case G is a free product of a free group and a finite cyclic group.

First we may assume that for each letter c occurring in r, $\sigma_c(r) > 0$. Suppose that for some a and b, $\sigma_a(r) > \sigma_b(r)$ and take $\alpha = \sigma_a(r)$ and $\beta = \sigma_b(r)$ as usual. Partition r as a product $s_0 a s_1 a s_2 \ldots a s_\alpha$ where each s_i has $\sigma_a(s_i) = 0$. By conjugating r we may as well assume that $s_0 = 1$. Then $\alpha > \beta = \sigma_b(r) = \sum_{i=1}^{n} \sigma_b(s_i)$ is a sum of α terms so some $\sigma_b(s_i) \leq 0$. By conjugating again we may as well assume that $\sigma_b(s_1) \leq 0$. But then $I_{a,b}(1, a, s_1 a \ldots s_\alpha) = 0$ and $I_{a,b}(a s_1, a, s_2 \ldots s_\alpha) = \beta - \sigma_b(a s_1)\alpha > 0$ and so this choice of a and b would be suitable for the above construction. Hence instead $\sigma_a(r)$ is the same for all a, say $\sigma_a(r) = m$. Next suppose that for some c both c and c^{-1} appear in r (still assumed to be cyclically reduced). Take a to be a letter such that $r = s_0 a s_1 a^{-1} s_2$ or else $r = s_0 a^{-1} s_1 a s_2$ with s_1 of minimal length (over all possible pairs of occurrences of letters and inverses). Say the former case holds and let b be a letter occurring in s_1. Then $\sigma_a(s_1) = 0$ since there can be no occurrence of a or a^{-1} in s_1, and $\sigma_b(s_1) \neq 0$ since not both of b and b^{-1} appear in s_1 by the minimality assumption. But then $I_{a,b}(s_0 a s_1, a^{-1}, s_2) = I_{a,b}(s_0, a, s_1 a^{-1} s_2) - \sigma_b(s_1)m$ and again a and b would work in the above construction.



$(\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}$

Hence each letter appears the same number of times with only positive exponents. If there exists a and b such that b does not occur or appears more than once between some pair of successive occurrences of a then these occurrences of a would become different y_i in the above construction. Hence if there are k generators and a is the first letter in r then each other generator must appear exactly once between the first and second occurrences of a in r, i.e. the second occurrence of a is the $k+1$-st letter of r. But then the second occurrence of the second letter of r must be the $k+2$-nd letter of r, etc. The relator r must be a power of the product of the generators appearing in r, and hence G is a free product of a free group (generated by the generators of G not appearing in r) and a group with presentation $\langle a_1, \ldots a_k : (a_1 \ldots a_k)^m \rangle$. But this last group is a free product of a free group of rank $k-1$ and $\mathbb{Z}_m$. Thus, in the case where no letter appears in r with zero exponent sum, either there exist letters a and b such that the above construction produces a one relator group H with relator r' of the same length but involving more generators such that $G * \mathbb{Z}$ is an HNN-extension of H, or else the relator r is of such a simple form that G is in fact a free product of a free group and a finite cyclic group. ∎

Proof of Lemma 5. Let X be a finite complex such that $\pi_1(X) = G$. Let Y be obtained from X by attaching a loop to X at a vertex $x \in X$. Then $\pi_1(Y) = G * \mathbb{Z}$. The universal cover $\tilde{Y}$ of Y is a union of copies of $\tilde{X}$, the universal cover of X, and copies of the real line $\mathbb{R}$. Let $p: \tilde{Y} \to Y$ be the covering projection. If a copy of $\tilde{X}$ intersects a copy of $\mathbb{R}$ it does so in exactly one point of $p^{-1}(x)$. If a copy of $\tilde{X}$ intersects a copy of $\mathbb{R}$ it does so in exactly one point of $p^{-1}(x)$. All copies of $\tilde{X}$ are mutually disjoint as are all copies of $\mathbb{R}$ (see figure).

Let $\tilde{X}_0$ be a copy of $\tilde{X}$ in $\tilde{Y}$. Let r, s: $[0, \infty) \to \tilde{X}_0$ be proper rays converging to the same end of $\tilde{X}_0$. It suffices to show that r and s are properly homotopic in $\tilde{X}_0$. Since $\tilde{Y}$ is semistable at infinity there exists $F: [0, \infty] \times [0, 1] \to \tilde{Y}$ a proper homotopy of r to s. If U is a component of $\tilde{Y} - \tilde{X}_0$ then $\bar{U}$, the closure of U in $\tilde{Y}$, is $U \cup \{v\}$ for some vertex $v \in p^{-1}(x)$. Since no proper ray in $\bar{U}$ converges to the same end of $\tilde{Y}$ as a ray in $\tilde{X}_0$, $F^{-1}(\bar{U})$ is compact. If a is any point in the topological boundary of $F^{-1}(\bar{U})$ then $F(a) \in \bar{U} - U = \{v\}$. Define $F': [0, \infty) \times [0, 1] \to \tilde{X}_0$ as follows. For $a \in F^{-1}(\tilde{X}_0)$ let $F'(a) = F(a)$. For $a \in F^{-1}(U)$ where U is a component of $\tilde{Y} - \tilde{X}_0$ and v the unique point of $\bar{U} - U$ let $F'(a) = v$. If C is a compact subset of $\tilde{X}_0$ let $U_1, \ldots, U_n$ be the components of $\tilde{Y} - \tilde{X}_0$ whose closures intersect C (finitely many since $C \cap p^{-1}(x)$ is finite). Then $(F')^{-1}(C) = F^{-1}(C) \cup \bigcup_{i=1}^{n} F^{-1}(\bar{U}_i)$. It follows that F' is proper and continuous. ∎

REFERENCES

1. R. Geoghegan and M. Mihalik: Free abelian cohomology of groups and ends of universal covers, *J. Pure and Appl. Algebra* **36** (1985), pp 123–137.
2. R. C. Lyndon and P. E. Schupp: *Combinatorial Group Theory*, Springer-Verlag, Berlin, Heidelberg, New York (1970).
3. M. Mihalik and S. Tschantz: Semistability of amalgamated products and HNN-extensions, *Mem. Am. Math. Soc.*, to appear.

Department of Mathematics,
Vanderbilt University,
Nashville,
Tennessee 37240,
U.S.A.